The Usefulness of the Impossible

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A straight line has no width, no depth, no wiggles, and no ends. There are no straight lines. We have ideas about these non-existent impossibilities: we even draw pictures of them. But they do not exist.

Ask a draftsman to draw a straight line. Place his product under a microscope. Observe the variable width; the darkness and lightness which mark its varying depth; the wiggles, both lateral and vertical, where the line plunges and climbs and wriggles among the fibers. Put the microscope away and observe the ends of the line where it runs off the paper. Finally, contemplate the curvature of space.

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A straight line hasn't even a definition. Heath, in his **Thirteen Books of Euclid's Elements**, discusses the problem. Before Euclid, Parmenides has stated "straight is whatever has its middle in front of both ends." Euclid defined a straight line as "a line which lies evenly with the points on itself." Heron, in the first century, A.D., suggested "a line stretched to the utmost." Equally old, although restated by Leibniz and put into the following form by Gauss, is:

> "The line in which lie all points that, during the revolution of a part of space about two fixed points, maintain their position unchanged . . ."

While this may seem definitions aplenty, the modern view as expressed by Pfleiderer is:

> "It seems as though the notion of a straight

> line, owing to its simplicity, cannot be explained by any regular definition which does not introduce words already containing in themselves, by implication, the notion to be defined, as though it were impossible, if a person does not already know what the term straight here means, to teach it to him unless by putting before him in some way a picture or a drawing of it."

A point has no dimensions, no existence, and no definition. A picture of a point is a ragged area of uncertain extent on a rough surface. By refining the picture, the area diminishes. Finally, we can refine no more. We place the portraits side by side in the sequence of successive refinement. Then we point far to the side and say, "The picture that belongs there, where refinement has been carried to the ultimate and the dimensions have entirely vanished; that picture, if it existed, would be a true dimensionless point."

A point has no dimensions, no existence, and no definition

Euclid lists twenty-three definitions which define more than twenty-three figments of the imagination. Next he postulates an ability to draw straight lines from point to point, to project these straight lines indefinitely in either direction, and to draw circles; all manifest impossibilities. He assumes all right angles are equal, although there are no right angles. And he includes the famous postulate of parallels, by denying which Riemann created the geometry of curved space. Lastly, Euclid introduces five "common notions" as axioms; that is, as self-evident truths; the very first of which is impossible, let alone true. "Things equal to the same thing are equal to each other." No two real things are precisely equal. The common equal is doubly doubtful. It is an impossible ideal which can be approached only imaginatively.

He who protests that this is a quibble, that for practical purposes equals do exist, merely impales himself on the other horn of a dilemma. In a chapter on Number in his **Aspects of Science**, Tobias Dantzig tells of two bars, A and B, so nearly the same length as to defeat all attempts to ascertain which is the longer. Practically, they are of equal length. Another bar, C, is so nearly the same length as B as to defy all attempts to show a difference. Practically, B and C are also of equal length. But when A is compared with C, there is no difficulty in proving that C is longer than A. So if we deal with practical equalities, things may be equal to the same thing without being equal to each other.

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The whole of geometry is consciously, willfully, deliberately antagonistic to reality. In classical geometry, the compass and straight edge are allowed, the ruler forbidden. The compass and straight edge are both mystical, for they produce true circles and lines. The ruler is a practical tool used by artisans and beneath the dignity of a Greek philosopher-mathematician. Modern geometry has exceeded the purity of the ancients. It deals with points which are not points but vague unspecified items; lines which are not lines but classes of items; and planes that are not planes but classes of classes. In some modern geometries, straight lines are distorted geodesies twisting and wriggling in a warped and changing space. In modern physics, these writhing monsters are chopped into a large, but finite, number of tiny, but not infinitesimal, discontinuous, discrete quanta. This is as near as we can get to "real" straight lines!

Nor is geometry the black sheep of mathematics. All

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mathematics carries the family taint; all mathematics is a gigantic tussle with nonexistent impossibilities. We are cautioned not to add poems and railroad trains, or to subtract centimeters from miles. Without referring to what is added on the balance sheets of business enterprises, let us merely note that if one ought not to add apples and bananas, one probably shouldn't add Jonathans and Rome Beauties, or big apples and little ones, or 1929 dollars and 1932 dollars. If we are to add at all, we must add unlikes, in violation of all mathematical regulations.

I shall not attempt to prove that mathematics is useful. I will admit it and so save myself the trouble of proving that here is a great and respected discipline where all is impossible and yet much is useful. The usefulness largely flows from the impossibility. Mathematical concepts have been simplified and generalized until they describe an imaginative world no part of which could possibly exist outside men's minds. But their simplicity and generality have made them amenable to the laws of logic. We can think about them with sufficient rigor to build a truly impressive edifice, much of which translates into physics and engineering.

II

Truth to the mathematician merely means freedom from internal inconsistencies. All mathematics begins with a set of axioms. Any set of axioms is as valid as any other as long as it avoids contradictory assumptions. Physics supposedly labors under the additional handicap of the experimental method. Its assumptions must be consistent with the readings of its meters and its gauges. The superstructure based on these assumptions must submit to experimental verification. As a result, the novice believes physics describes objective reality. Only mathematics enjoys a greater reputation for the profundity and pervasiveness of its Truths.

Physics, too, is plagued by questionable tactics. Laws proved untrue are easily rescued by adding terms. For example, Boyle's law was found untrue for high pressures and low temperatures. Van der Waal argued that it held precisely only for a perfect gas with point molecules; that as the molecules of a real gas crowded closer together the error became more noticeable. He saved Boyle's law by introducing another term to take care of the size of the molecules. If more refined experiments reveal further discrepancies, the law may be rescued again by introducing corrections for the velocities of the molecules, or for their nationalities. If the velocities won't explain the hypothetical inaccuracies of the law, the physicist may try accelerations or differential equations of still another order. If a proportional law doesn't fit, he can try inverse proportions, or squares, or exponents. Somewhere he can find a physical measurement which seems to have some kind of mathematical relation to the observed deviations, and all such troubles will surely yield to the same treatment.

Newton's laws of dynamics exhibit a refinement of the technique. Nowhere has anyone ever seen a body continue moving in a straight line with uniform velocity. Nor has anyone ever seen a body at rest remain at rest. Indeed, we do not even know what the words "at rest" mean. I quote Dantzig loosely and out of context:

Indeed, we do not even know what the words "at rest" mean

> "How can a bird fly in a straight line and at constant speed in the teeth of gravity? The answer is that the resistance of the air balances the gravitational pull. How can a ball

> roll down an inclined path at constant velocity instead of constantly accelerating? The friction of the surface accounts for this. Why do the particles of a solid body stay put, instead of flying asunder under the action of gravity? Cohesive internal forces keep them together. Whenever and wherever a violation of the principle of inertia is observed, it is sufficient to invoke some reaction to have the difficulty vanish, as though by magic."

The accountant has a nasty name for the technique. He calls it "plugging" to force a balance.

From plugging the accounts, it is only a short step to the next refinement. The physicist avoids the need of a rescuing plug by making his laws true by definition. I quote Poincaré's great work, **Science and Hypothesis**:

> "The principles of dynamics at first appeared to us as experimental truths; but we have been obliged to use them as definitions. It is by definition that force is equal to the product of mass by acceleration; here, then, is a principle which is henceforth beyond the reach of any further experiment. It is in the same way by definition that action is equal to reaction."

Having set up his definitions, the physicist calibrates his instruments accordingly. Having defined force as proportional to acceleration, and having chosen some force as a unit, he doubles the acceleration and marks his force meter two at the point indicated. Ever after, whenever he measures with the instruments so created, his findings bear out his definitions; his laws become absolutely true.

The culmination of the technique is the creation of so anthropomorphic a cosmology as to be beyond the ability of men to prove or disprove it. Eddington states the case thus:

> "We have found a strange footprint on the shores of the unknown. We have devised profound theories, one after another, to account for its origin. At last we have succeeded in reconstructing the creature that made the footprint. And lo! it is our own."

Dantzig goes farther. I quote at length:

> "For however phantastic a universe our mind may conceive, our mind can also conceive it peopled by species, endowed with consciousness, intelligence and mobility, which in the course of time would arrive at a cosmology identical with our own.
>
> "Seeking permanence in the shifting chaos of their perceptions, these beings would eventually discover in their environment bodies which would behave in relative unison to their own. Accepting these bodies for rigid standards, they would proceed to survey and measure the universe with their aid. Singling out some cyclic phenomena which recur in relative synchrony to each other, and to their own physiological processes, these beings would finish by identifying these temporal series with their own stream of consciousness. Convinced that their universe was independent of their consciousness, they would affirm the objective character of their conception of time, and proceeding beyond the narrow confines of their own experience, they would extend their conception to the world at large, conceiving the latter as floating with absolute uniformity on the stream of duration.

> And transferring to their universe their own physiological and psychological attributes, they would fill space with bristling forces and shackle history to a causal chain."

As with mathematics, I propose to assume that physics is useful, although I feel some doubt has been cast upon its objective validity. The rigorous exclusion of all non-measurable phenomena, and the careful formulation of its definitions and axioms as the calibrations of the instruments to be used in the experimental verification of physical laws, have simplified and generalized physics along the lines of the mathematical model. This has added immensely to its precision, to its power, and to its usefulness.

III

Mathematics and physics are theoretical. Let us turn from abstraction and generalization to practical application. Engineering is as riddled with impossibility as mathematics or physics. Of course, engineering is full of mathematics and physics; they are the basic sciences. But I do not rest my case here. Engineering data are as impossible as engineering's mathematical method. Engineering data are average values, usually treated in engineering calculations as absolutes. According to Mills' **Materials of Construction**, structural steel has an elastic limit of 35,000 pounds per square inch, a tensile strength of 65,000 pounds per square inch, and a modulus of elasticity of 30,000,000 pounds per square inch. No standard deviations are given. Such are found only in the inexact, semi-scientific disciplines of biology, psychology and economics.

In calculating the distortion of bridge members, the engineer implicitly assumes constant cross-sections, uniform crystal structures, and homogeneous chemical composition from end to end of each beam. Anyone who has seen the scale peel off an ingot as it goes through the rolls knows the constant cross-section is a crude fiction. Heat treatment and the working of steel so change crystal structures as to make the assumption of uniformity in ordinary rolled beams heroic indeed.

However, the engineer is a practical fellow. While his equations assume a 35,000-pound elastic limit in a perfectly uniform beam, he does not. To keep his bridges from falling when these assumptions err on the wrong side, he typically designs them to carry seven times the expected maximum load. This makes bridges expensive but safe. The engineer can boast that they seldom fall. Yet engineers are modest braggarts. The multiplier used to assure safety has been rechristened the "factor of ignorance."

IV

In mathematics, physics, and engineering, we see that the impossible may be useful at least as a calculating device.

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